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On the existence of monopoles in grand unified theories

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Abstract. Let G be a compact and connected Lie group. A grand unified theory having G as gauge group can admit monopole solutions only if a topological constraint is satisfied.

It has been argued recently (McInnes 1984) that 'an electroweak gauge theory based on a compact connected non-semisimple Lie group admits no stringless magnetic monopoles, except in the case in which the charge operator lies entirely inside the semisimple part of the algebra'.

The proof is based on the assumption that the unbroken theory is defined on a *trivial* 'unifying' bundle P. Over the 'two-sphere at infinity', this bundle is reduced to a U(1) bundle $(Q, \mathrm{U}(1))$. To get a monopole, this reduced bundle must be *non-trivial.* McInnes shows then that if the condition above is *not* satisfied, any reduction of P is *trivial* and so there are no monopoles.

The example below shows however, that the reduced bundle can be trivial, even if the electromagnetic direction belongs to the semisimple part. In proposition 1, theorem 2 we give instead the *necessary and sufficient condition* for getting a non-trivial reduction from a trivial unifying bundle. Our results are valid for an *arbitrary* (and not only for electroweak) *grand unified gauge theory* (GUT).

Let G be a compact and connected Lie group and consider a GUT having G as gauge group. In geometric language, a monopole is described by a connection form A on a trivial 'unifying' bundle $P = R^3 \times \mathrm{G}$, and an equivariant map Φ on P which takes its values in a suitable representation space. Choosing a global trivialisation of P, Φ is identified with the physical *Higgs field.*

Spontaneous symmetry breaking means that over S^2, the 'two-sphere at infinity', Φ takes its values in an orbit G/H of G, where H is a closed subgroup of G.

Geometrically, the restriction of P to S^2 (which we denote also by P) must *reduce* to an *H bundle* (Q, H). This reduction is defined by Φ. (To have finite energy one requires that $\mathrm{D}\Phi = 0$ over S^2. This implies that the Yang-Mills connection also reduces to (Q, H)).

The isomorphism class of a principal bundle (Q, H) is characterised by the fundamental topological invariant $[Q] \in \pi_1(\mathrm{H})$, the image under the injective homomorphism δ: $\pi_2(\mathrm{G/H}) \to \pi_1(\mathrm{H})$ of the homotopy class $[\Phi] \in \pi_2(\mathrm{G/H})$.

As proved in Horváthy and Rawnsley (1985c)—see also (Bais 1981)—if P is a G bundle, an H bundle $Q \subset P$ is a reduction of P if and only if $i_*[Q] = [P]$, where i_*

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is the group homomorphism $i_*: \pi_1(\mathrm{H}) \to \pi_1(\mathrm{G})$ induced by the inclusion map $i: \mathrm{H} \hookrightarrow \mathrm{G}$. Applied to the trivial bundle P whose class $[P]$ vanishes, we get the following proposition.

Proposition 1. The H bundle Q is the reduction of a trivial unifying bundle (P, G) over S^2 if and only if

$$i_*[Q] = 0. \qquad (1)$$

If G is simply connected, (1) is automatic.

This condition can be translated to *algebraic* terms. First, remember that over S^2 the Yang-Mills-Higgs equations reduce to an $\mathscr{k}$-valued pure Yang-Mills equation

$$\mathrm{D}^* F = 0 \qquad (2)$$

whose solution has been found by Goddard *et al* (1977): these exists a fixed, quantised (i.e. $\exp(4\pi\Pi) = 1$) vector Π—the so-called '*non-Abelian charge*'—such that

$$F = F_{\mathrm{D}} \cdot \Pi \qquad (3)$$

where F_{D} is the canonical surface form of S^2. Π is unique up to constant gauge transformation. Thus Π can always be chosen to belong to any given Cartan algebra.

Physically, F_{D} is the field strength tensor for a Dirac monopole of magnetic charge $1/2e\hbar$.

The homotopy invariant $[Q] \in \pi_1(\mathrm{H})$ is represented by the loop

$$\exp(4\pi t\Pi) \qquad 0 \leqslant t \leqslant 1. \qquad (4)$$

Such a solution is called a *monopole* if (4) has non-vanishing homotopy class in $\pi_1(\mathrm{H})$.

Next, in Horváthy and Rawnsley (1985a) we have shown that, for any compact and connected G,

$$\pi_1(\mathrm{G}) \simeq \pi_1(\mathrm{G})_{\mathrm{free}} + \pi_1(\mathrm{K}) \qquad (5)$$

where K is the semisimple subgroup of G generated by the derived algebra $\mathscr{k} = [\mathscr{G}, \mathscr{G}]$.

To describe the *free part*, let $\tilde{\Gamma} = \{\xi \in \mathscr{G} \,|\, \exp(2\pi\xi) = 1\}$ be the unit lattice of G, and denote $z: \mathscr{G} \to Z(\mathscr{G})$ as the projection of the Lie algebra of G onto its centre. $z(\tilde{\Gamma})$, the image of $\tilde{\Gamma}$ under z, is a lattice in $Z(\mathscr{G})$. According to § 2 of (Horváthy and Rawnsley 1985a) there is an isomorphism ρ of $\pi_1(\mathrm{G})_{\mathrm{free}}$ with the *lattice* $z(\tilde{\Gamma})$, considered as a free Abelian group $\simeq Z^p$, where p is the dimension of $Z(\mathscr{G})$. (Choosing a Z basis for $z(\tilde{\Gamma})$, we get p 'quantum' numbers $m_1, \ldots, m_p$). For a monopole given by a non-Abelian charge Π, $\rho[Q] = 2z(\Pi)$.

$\pi_1(\mathrm{K})$—a *finite Abelian group*—is described as follows: Let $\mathrm{T} \subset \mathrm{K}$ be a maximal torus with Lie algebra t, and let K^* be the simply connected group having $\mathscr{k}$ as Lie algebra. Then we have two unit lattices, $\Gamma = \{\xi \in t \,|\, \exp(\mathrm{K}2\pi\xi) = e\}$ and $\Gamma^* = \{\xi \in t \,|\, \exp(\mathrm{K}^* 2\pi\xi) = e^*\}$. $\Gamma^* \subset \Gamma$ and it is known (Wallach 1973) that $\pi_1(\mathrm{K}) \simeq \Gamma/\Gamma^*$.

The unit lattice Γ can be described in terms of the *roots* (Humphreys 1972) of the simply connected group K^*. Let Δ be the set of roots of $(\mathscr{k}^{\mathrm{C}}, t^{\mathrm{C}})$, so $\Delta \subset \mathrm{i}t^*$ (algebraic dual). Let Δ_+ be the set of positive roots. If $\alpha_1, \ldots, \alpha_r$ are the simple roots then there are elements $H_{\alpha_i} \in t^{\mathrm{C}}$ with

$$B(\mathrm{H}_{\alpha_i}, \mathrm{H}) = \alpha_i(\mathrm{H}) \qquad \mathrm{H} \in t$$

where B is the (non-degenerate) Killing form of the semisimple algebra $\mathscr{k}$. Γ^* is generated over Z by $\sqrt{-1}\mathrm{H}_1, \ldots, \sqrt{-1}\mathrm{H}_r$, where $\mathrm{H}_i = 2\mathrm{H}_{\alpha_i}/\alpha_i(\mathrm{H}_{\alpha_i})$ (Wallach 1973).

In order to reformulate proposition 1 in terms of the non-Abelian charge, let us decompose Π as

$$\Pi = z(\Pi) + \Pi' \tag{6}$$

where $\Pi' \in \mathscr{k}$. Our remarks prove the following theorem.

Theorem 2. The solution of the asymptotic field equations (1) defined by a quantised vector $\Pi \in \mathscr{h}$ is a monopole solution of the spontaneously broken GUT iff

$$z_G(\Pi) = 0 \tag{7a}$$

$$2\Pi' \in \Gamma_G^* \tag{7b}$$

either

$$z_H(\Pi) \neq 0 \qquad \text{or} \qquad 2\Pi' \notin \Gamma_H^* \tag{8}$$

where the subscript G (respectively H) refers obviously to the groups G (respectively H).

Indeed, equations $(7a)$-$(7b)$ mean that (Q, H) is a reduction of the trivial unifying bundle P whilst (8) ensures that (Q, H) has already non-trivial topology.

Conversely, assume that we are given a Π satisfying the conditions $(7a)$-$(7b)$ and (8). Are we able to build a monopole out of it? The answer is yes, at least asymptotically.

To see this, consider first a closed subgroup H or G, and consider a principal H bundle Q. H acts on $Q \times \mathrm{G}$ according to $(q, g)h = (qh, h^{-1}g)$. The associated bundle $P = Q \times_{\mathrm{H}} \mathrm{G}$ is a principal G bundle: G acts on P according to $[q, g]g' = [q, gg']$. Furthermore, Q is a reduction of this bundle, since it can be identified with $\{[q, e] | q \in Q\}$.

If A is a connection on Q, it has a unique extension to P: indeed, $(A + g^{-1}\,\mathrm{d}g)$ is an H-invariant $\mathscr{G}$-valued 1-form on $Q \times \mathrm{G}$; if $\tilde{\xi}$ is the vector field on $Q \times \mathrm{G}$ defined by the infinitesimal action of $\xi \in \mathscr{h}$, then $\tilde{\xi} \lrcorner (A + g^{-1}\,\mathrm{d}g) = A(\hat{\xi}) - \xi = 0$ since $\hat{\xi}$ is the fundamental vector field corresponding to ξ. This extended connection obviously reduces to Q.

Let us return to our original problem. $2\Pi = n \cdot \xi_0$ for some integer n and a minimal U(1) generator ξ_0 since 2Π is quantised. For each $n \in Z$ there is a unique U(1) bundle Y_n over S^2—the well known Hopf bundle—with connection form $\mathrm{i}\alpha_n$ and curvature in F_{D}. F_{D} is harmonic, $(\mathrm{d}F = 0$ and $\mathrm{d} * F = 0)$ and hence (Y_n, α_n) is a solution of the U(1)-Yang-Mills—i.e. the Maxwell—equations. This is furthermore the unique solution up to equivalence.

Let $\mathrm{U} = \{\exp(2\pi t\xi_0) | 0 \leq t \leq 1\}$ be the U(1) subgroup of H generated by ξ_0. Then Y_n can be viewed as a principal U bundle, and, as demonstrated above,

$$Q = Y_n \times_{\mathrm{U}} \mathrm{H} \tag{9}$$

is a principal H bundle over S^2 with a U connection $\alpha_n \cdot \xi$ and with curvature $nF_{\mathrm{D}} \cdot \xi_0 = F_{\mathrm{D}} \cdot 2\Pi$.

The extended connection A solves the H-valued Yang-Mills equations (2) since the field equations are gauge invariant and on Y_n the Maxwell equations are satisfied.

To have a local picture, choose the standard covering $V_+ = S^2 \backslash \{\text{south pole}\}$ and $V_- = S^2 \backslash \{\text{north pole}\}$ and select 1-forms $\alpha_\pm$ with $F_{\mathrm{D}} | V_\pm = \mathrm{d}\alpha_\pm \cdot Y_n$ admits then sections s_+ and s_- with $s_\pm^* \alpha_n = n\alpha_\pm$ in $V_+ \cap V_- \cdot s_+ = s_- h$ with $h: S^1 \to \mathrm{U}$; in fact $h(\theta) = \exp(n \cdot \xi_0 \theta) = \exp(2\pi\theta)$, where $0 \leq \theta \leq 2\pi$, parametrises the equatorial circle. $h(\theta)$ is the transition function also for Q, so the fundamental topological invariant $[Q] \in \pi_1(H)$ which characterises Q is the homotopy class of the loop (4). The extended connection is expressed in these gauges as

$$(s_\pm^* A)_{s(x)g} = \mathrm{Ad}\, g^{-1} n\alpha_\pm + g^{-1}\, \mathrm{d}g, \tag{10}$$

This same procedure can be applied once more to get the principal G bundle

$$P = Q \times_H G \tag{11}$$

with the extended connection. Q is then a reduction of P, which, by theorem 2, is a trivial bundle.

(P, G) trivial implies that there exists a global section $\gamma: S^2 \to Q \times_H G$. However, such a section is equivalent to an H-equivariant function (which we denote also by γ) on Q with values in the fibre, G: $\gamma(qh) = h^{-1}\gamma(q)$. But $Q \times_H G$ contains the H-bundle Q as sub-bundle, so Q is also a reduction of $Q \times H$ by a Higgs field $\Phi: S^2 \to G/H$. In fact

$$\Phi(x) = [\gamma(q)]^{-1} H. \tag{12}$$

(12) is convariantly constant since the G connection reduces to (Q, H). We summarise with the theorem below.

Theorem 3. Any quantised $\Pi \in \mathscr{h}$ which satisfies (7*a*)-(7*b*) and (8) defines an asymptotic monopole configuration, constructed as above.

The same conclusions can be reached of course without using fibre bundles. In Goddard and Olive (1978) and (Coleman 1983) it is proved in fact that monopoles can exist only if the non-Abelian charge lies in the kernel of i_*. (7*a*)-(7*b*) is just the algebraic translation of this requirement.

The situation studied by McInnes (1984) corresponds to $H = U(1)$ generated by a minimal generator ξ_0. H is obviously identified with the gauge group of electromagnetism. In this case Π must be parallel to ξ_0, $\Pi = n \cdot \xi_0$ for some integer n, since π must belong to the Lie algebra of H and Π is quantised. (7*a*) requires, as stated by McInnes, that the charge operator lie entirely in the semisimple part of the Lie algebra of G, so no monopole solution exists if this is violated. (7*b*) however is an *additional* condition. ((8) is automatic for $K = U(1)$.)

In the *Weinberg-Salam* model $G = U(2) = [U(1) \times SU(2)]/Z_2$; $\mathscr{G} = u(1) + su(2)$. $K = SU(2)$ is simply connected and so $\Gamma = \Gamma^*$. (7*b*) is thus automatic. To get monopoles, Π must be chosen in the su(2) part. (Such a choice is however unphysical, since the direction of the residual U(1) has, according to experience, Weinberg angle $\sin^2 \theta_W \simeq 0.23$).

Consider instead an electroweak theory based on $G = U(1) \times SO(3)$ whose Lie algebra is the same as that of the Weinberg-Salam model, $\mathscr{G} = u(1) + so(3) \simeq u(1) + su(2)$. To get a monopole, the non-Abelian charge vector 2Π must be chosen in Γ^* of SO(3). Take

$$T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad 0 \leq \theta \leq 2\pi.$$

with Lie algebra

$$t = \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad a \in R.$$

Let

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad L_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We have $B(X, Y) = \mathrm{Tr}(XY)$ and $[L_3, L_1 \pm \mathrm{i}L_2] = \mathrm{i}(L_1 \pm \mathrm{i}L_2)$.

If α on t is $\alpha(aL_3) = -\mathrm{i}a$ then the roots are $\pm\alpha$. Thus α is a simple root and $\mathrm{H}_\alpha = aL_3$ and consequently $\mathrm{Tr}(\mathrm{H}_\alpha X) = \alpha(X)$, so that $\mathrm{Tr}(aL_3^2) = a(L_3) = -\mathrm{i}$, and thus $a = -\mathrm{i}/2$. Hence $\mathrm{H}_\alpha = \mathrm{i}\mathrm{H}_3/2$, whilst $\alpha(\mathrm{H}_\alpha) = \frac{1}{2}$, so $\mathrm{H}_1 = 4\mathrm{H}_\alpha = 2\mathrm{i}L_3$. Thus Γ^* is generated by $2L_3$. Γ is obviously generated by L_3. (Thus $\Gamma^* = 2\Gamma$ and so $\pi_1(\mathrm{SO}(3)) = Z_2$ as expected.) According to theorem 2 we can get monopoles in this theory if and only if the non-Abelian charge vector has the form,

$$\Pi = \Pi' = n \cdot L_3 \tag{13}$$

for some *integer n*. The condition of McInnes would give monopoles also for *half-integer n*.

How did McInnes miss the condition (7*b*)? He uses characteristic classes (Kobayashi and Nomizu 1969). However, he considers only primary invariants—which depend only on the *free part* of π_1 (Horvàthy and Rawnsley (1985a). The problem studied here is related to the fate of GUT monopoles under successive symmetry breakings (Bais 1981, Horvàthy and Rawnsley 1985b).

References

Bais F A 1981 *Phys. Lett.* **98B** 437
Coleman S 1983 *Proc. E. Majorana Summer School, Erice '81* ed A Zichichi (New York: Plenum) p 297
Goddard P, Nuyts J and Olive D 1977 *Nucl. Phys.* B **125** 1
Goddard P and Olive D 1978 *Rep. Prog. Phys.* **41** 1357
Horvàthy P A and Rawnsley J H 1985a *Commun. Math. Phys.* **99** 517
—— 1985b *Lett. Math. Phys.* to appear
—— 1985c *Commun. Math. Phys.* submitted for publication
Humphreys J E 1972 *Introduction to Lie Algebras and Representation Theory* (Berlin: Springer)
Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* vol 1 (New York: Interscience)
—— 1969 *Foundations of Differential Geometry* vol 2 (New York: Interscience)
McInnes B T 1984 *J. Phys. A: Math. Gen.* **17** 3287
Wallach N R 1973 *Harmonic Analysis on Homogeneous Spaces* (New York: Marcel Dekker)